Multiple description lattice vector quantization using multiple A₄ quantizers

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Abstract: This paper presents a new Multiple Description Lattice Vector Quantization (MDLVQ) based on A₄ lattice quantizers for multiple description coding abbreviated as MDLVQ-A₄. The Coinciding similar A₄ sublattices are used to quantize the input stream for multiple description coding system. The use of multiple coinciding quantizers eliminates the labeling function of the traditional schemes. Experimental results of MDLVQ-A₄ scheme for image coding show a higher performance in terms of PSNR of the side decoders as compared to the renowned techniques of image coding for several test images.

Keywords: Lattice Vector Quantization (LVQ), Multiple Description (MD) Coding, Wavelet transforms.

Classification: Science and engineering for electronics

References

1. Introduction

In Multiple Description (MD) coding the input data are encoded into several independent descriptions prior sending them over different channels. At the receiver, if all the descriptions are received correctly, the original data can be reconstructed accurately. However if some of the descriptions get lost, the rests are used in the side decoders to find the estimate of the original data.

In the first work on MD coding different scalar quantizers are used to encode the input data into two different descriptions [1]. The design of multiple descriptions coding that utilizes lattice vector quantizer (MDLVQ) is first described in [2]. The application of MDLVQ in image coding together with wavelet transformation is first presented in [3].

In this paper, a new MD coding scheme based on the coinciding A₄ lattice quantizers (MDLVQ-A₄) is presented. The coinciding similar sublattices of A₄ are geometrically similar sublattices of A₄ lattice generated by different generator matrices. The new scheme is different from the scheme presented in [2] and [3] because it does not include the labeling function. In addition the MDLVQ-A₄ scheme utilizes different sublattices rather than a single sublattice as in [2]. Additionally, in the MDLVQ-A₄ scheme, the sublattices have the same index, whereas the sublattices in [3] and [5] utilize different indices.

The advantage of using multiple coinciding sublattices in MDLVQ-A₄ is that they can be used to represent data with descriptions that can serve as good approximations for the original data. Simulation results of the MDLVQ-A₄ scheme with several standard test images show an average of 6dB increments of the PSNR in the side decoders as compared to the schemes presented in [3] and [5] with bit rate in the range of [0.25 1] bpp.

The rest of this paper is organized as follows; Section 2 presents the background of MDLVQ-A₄. Section 3 defines the coinciding similar sublattices. Section 4 describes the calculation of correlation between lattices. Section 5 presents the results and Section 6 concludes the paper.

2. Geometrically similar Sublattice of A₄

A lattice Γ is a subset of points in the Euclidean space which are selected using a generator matrix. For example the root lattice A₄ = \{x₀, x₁, x₂, x₃, x₄ \in \mathbb{Z}^5 : \sum x_i = 0\} is generated by

\[
M_{A_4} = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}
\]  

(1)

The associated Gram matrix of \(M_{A_4}\) is given by:

\[ MM' = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad (2) \]

It is difficult to find the similar lattices of \( A_4 \) using the description of \( A_4 \) as a lattice in a 4-dimensional hyperplane of \( \mathbb{R}^5 \) \([7, 8]\). Therefore the description in \( \mathbb{R}^4 \) space is usually used:

\[ L = 1/2 \langle (2,0,0,0), (-1,1,1,1), (0,-1,0,0), (0,1,-\sigma,-\tau) \rangle_2 \quad (3) \]

where \( \tau = (1 + \sqrt{5})/2 \) and \( \sigma = (1 - \sqrt{5})/2 \). The Gram matrix of \( L \) is \( 1/2 MM' \). It indicates that \( L \) is a scaled copy of the root lattice \( A_4 \) \([7]\).

This description enables us to employ the quaternion algebra, based on the inclusion of a root system of type \( A_4 \) within a type of \( H_4 \) \([7]\). The Hamiltonian quaternion algebra \( \mathbb{H}(K) \) is defined

\[ \mathbb{H}(K) = \{ q = (a + bi + cj + dk) : a, b, c, d \in K \} \quad (4) \]

where \( K \) is the golden quadratic field. The generated elements of \( \mathbb{H}(K) \) satisfy the Hamilton’s relations \( (i^2 = j^2 = k^2 = ijk = -1) \). In \( \mathbb{H}(K) \), there are two important mapping schemes i.e. the conjugation map, \( q \to \overline{q} = (a - bi - cj - dk) \) and the twist map, \( q \to q' := (a', b', c', d') \) where ‘ \( \overline{q} \) ’ is the algebraic conjugation in \( K \), defined by \( \sqrt{5} \to -\sqrt{5} \) \([7]\).

The choice of \( L \) among many realizations of the root lattice \( A_4 \) is motivated by the observation that \( L \) is a subset of the so-called icosian ring \( \mathbb{I} \) \([7]\). The icosian group is a multiplicative group of order 120 consisting of the quaternions

\[ 1/2 \langle \pm 2, 0, 0, 0 \rangle^A, 1/2 \langle \pm 1, \pm 1, \pm 1 \rangle^A, 1/2 \langle 0, \pm 1, \pm \sigma, \pm \tau \rangle^A \quad (5) \]

where \( (a, b, c, d) \) means \( q = a + bi + cj + dk \), and superscript \( A \) means that all even permutations of the coordinates are permitted\([9]\). The coordinates belong to the golden quadratic field \( K = \mathbb{Q}(\sqrt{5}) = \{ a + b\sqrt{5} : a, b \in \mathbb{Q} \} \) \([9]\). The icosian ring \( \mathbb{I} \) is the set of all finite sums \( q_1 + \cdots + q_n \), where each \( q_i \) is in the icosian group. An element \( p \in \mathbb{I} \) is called \( I \)-primitive if \( \alpha p \in \mathbb{I} \) with \( \alpha \in \mathbb{Q} \) \([6]\).

The similar sub lattices of \( A_4 \) are in one-to-one relation to those of the lattice \( L \). If \( p \in \mathbb{I} \) then \( p \mathbb{I} p \subset L \) \([7]\). This means that all similar sub lattices (SSL) of the lattice \( L \) are images of \( L \) under orientation preserving mappings of the form \( x \to \alpha px\bar{p} \), with \( p \in \mathbb{I} - \text{primitive} \) and \( \alpha \in \mathbb{Q} \) \([7]\). Thus we can find 120 generator matrices using the 120 icosian elements. In fact, the icosian ring, as listed in \([9]\), consists of 60 different elements of the form \( \pm p \).

As a consequence, there are 60 pairs of generator matrices available with the same Gramm matrix. As an example consider the icosian \( p_{34} = 1/2 \langle -1, 0, -\tau, \sigma \rangle \).

Its corresponding twisted map is \( \tilde{p}_{34} = 1/2 \langle -1, \tau, -\sigma \rangle \). The generator matrix associated to \( p_{34} \) is \( L_{34} \) and its Gramm matrix associated is the same as \( A_4 \).

\[ L_{34} = p_{34} L_{\tilde{p}_{34}} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 0 & +\tau & +\sigma \\ 0 & 1 & +\sigma & +\tau \\ 0 & -2 & 0 & 0 \end{bmatrix} \quad (6) \]

3. Multiple quantizers design

In the MDLVQ-\( A_4 \) scheme the input stream is quantized by multiple lattice vector quantizers that use the root lattice \( A_4 \) as their codebooks. As described in
Section 2, there are 60 pairs of generator matrices that can be used to generate the $A_4$. These generator matrices are similar sublattices of $L$. Therefore they are all in $\mathbb{R}^4$ space. There is a one-to-one relation between these generator matrices in $\mathbb{R}^4$ and $M_{A_4}$. Thus, we can map each of them to $M_{A_4}$ using an appropriate transformation matrix. For example $T_{34}$ projects $L_{34}$ to $M_{A_4}$ and $1/2T_{34}^T$ to maps from the $M_{A_4}$ back to $L_{34}$.

$$T_{34} = \frac{1}{\tau - \sigma} \begin{bmatrix} 0 & 0 & 0 & \tau - \sigma & \sigma - \tau \\ 0 & \sigma - \tau & \tau - \sigma & 0 & 0 \\ -\tau & -\tau & 2 & -\sigma & -\sigma \\ \sigma & \sigma & -2 & \tau & \tau \end{bmatrix}$$ (7)

In this work a modified version of the fast quantizing algorithm [9] is used to implement different lattice quantizers. The Fast quantizing algorithm includes 6 steps. In step 1 the n-D input must be transformed to n+1-D version according to the definition of the $A_n$. In step 2 the 5-D input vectors are projected onto the closest point on the 5D hyperplane $\sum x_i = 0$. In this work step 2 is modified in order to produce different quantizers. The steps 3, 4 and 5 are followed the same as the fast quantizing algorithm. In the last step, knowing that which similar sublattice of the lattice $L$ is used, an appropriate transformation matrix is used to map the 5-D quantized input back to the 4-D version. If $L_{34}$ is selected then the appropriate transformation matrices $T_{34}$ and $1/2T_{34}^T$ will be used. Thus it is possible to have at least 60 different descriptions for a single input stream of data.

As shown in Fig.1, the labeling function is eliminated from the MDC system. The elimination of the labeling function decreases the imposed distortion in side decoders because the excess distortion term which is affected by the labeling function, is eliminated. In addition the computational burden is decreased. However, it has a side effect on the encoding performance i.e. the size of the codebook becomes too large. Thus higher bit rates are required to encode the streams. In order to alleviate this side effect we have normalized the wavelet coefficients before sending them to the quantizers.

4. The new MDLVQ-$A_4$ for image coding

The application of the new MDLVQ-$A_4$ for image coding is presented in this section. In theory it is possible to reduce the quantization error by increasing the number of lattice quantizers up to 60. However in practice it is neither possible nor efficient to design such a scheme and according to our experiments the quantization error is not completely eliminated. In fact, all these lattice quantizers do not necessarily produce descriptions that are similar enough so that they can be used in the MDC scheme. In order to investigate the feasibility of increasing the number of quantizers, the correlation between 60 sample lattices generated by the 60 generator matrices are calculated in Section 5. At the moment suppose that the lattices generated by the icosians $p_{1}$, $p_{21}$, and $p_{34}$ show very high correlations and they are used in the new MDLVQ-$A_4$ scheme.

The proposed MDLVQ-$A_4$ scheme consists of several different modules with different functionalities. First, the input image is transformed using the 10/18 Daubechies wavelet with 4 levels of decompositions as in [3] and small
coefficients are set to zero. Then the coefficients are normalized to cope with the transmission rate constraints. In LVQ1, LVQ21 and LVQ34 the input stream vectors are quantized using multiple lattice quantizers of $\Lambda_k$. The quantized streams are encoded using a basic zero-order arithmetic codec before being sent over the channels.

At the receiver the reverse processes are applied. If all the descriptions are received correctly then the central decoder finds an approximate of the original point. In the central decoder the 5-D input descriptions are mapped back to the 4-D version using the appropriate transformation matrices $1/2T_1$, $1/2T_{21}$ and $1/2T_{34}$. Then, the average of the three 4-D vector is calculated as the approximate value of the original input. If any of the descriptions is missing, the proper side decoder and suitable transformation matrices are used to find the approximation point. The new produced descriptions all offer very good approximations of the input data.

![Figure 1](image)

Figure 1 The new MDLVQ-A4 scheme consists of: Wavelet transformer, Vectorization, LVQs, arithmetic encoders, channels, arithmetic decoding and the decoders.

### 5. Calculating the correlation between lattices

It is not possible to find the correlation between lattices with infinite number of entries. Therefore, we have to do the experiments on sample lattices. In order to generate the sample lattices, first a matrix including all 4-D permutation of the integer values ranging from -5 to +5 is produced. Second, 60 sample lattices are generated by multiplication of the permutation matrix with the 60 generator matrices. In the third step, the average mutual one to one distances between the sample lattices are obtained as a metric of correlation.

In order to find the distance between two sample lattices, $I$ and $J$, for every lattice point in $I$, the nearest point in $J$ is found. Then, the distance between these two points is obtained. The average of these partial distances is considered as the distance between lattice $I$ and lattice $J$. In mathematical this is expressed:

$$\text{Dist}(I, J) \equiv \frac{1}{n} \left( \sum_{m=1}^{n} \min\{d(I_m, J_1), \ldots, d(I_m, J_n)\} \right).$$

(8)

where $n = 11^4$, and $d(I_m, J_1)$ is the distance of points $I_m$ and $J_1$.

An immediate result of the distance measure analysis is that every sample lattice has a pair that their relative distance is very close to zero i.e. they are coinciding lattices according to the selected range of integers. Thus the 60 lattices are grouped into 30 pairs and presented in Table 1. Each of these pairs can be used to design a two channel coinciding MDC system. In the proposed MDLVQ-A4 scheme three pairs of $(L_1, L_{21}), (L_1, L_{34})$ and $(L_{21}, L_{34})$ are used in side decoders.
and the triple of \( (L_4, L_{21}, L_{34}) \) is used in the central decoder.

**Table 1 The 30 pairs of the coinciding lattices**

<table>
<thead>
<tr>
<th>((L_4, L_{21}))</th>
<th>((L_2, L_{49}))</th>
<th>((L_3, L_{38}))</th>
<th>((L_4, L_{38}))</th>
<th>((L_5, L_{42}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((L_6, L_{25}))</td>
<td>((L_7, L_{22}))</td>
<td>((L_8, L_{59}))</td>
<td>((L_9, L_{46}))</td>
<td>((L_{10}, L_{45}))</td>
</tr>
<tr>
<td>((L_{11}, L_{14}))</td>
<td>((L_{12}, L_{57}))</td>
<td>((L_{13}, L_{56}))</td>
<td>((L_{15}, L_{43}))</td>
<td>((L_{16}, L_{23}))</td>
</tr>
<tr>
<td>((L_{17}, L_{55}))</td>
<td>((L_{18}, L_{50}))</td>
<td>((L_{19}, L_{44}))</td>
<td>((L_{20}, L_{37}))</td>
<td>((L_{41}, L_{58}))</td>
</tr>
<tr>
<td>((L_{35}, L_{60}))</td>
<td>((L_{24}, L_{27}))</td>
<td>((L_{26}, L_{53}))</td>
<td>((L_{29}, L_{40}))</td>
<td>((L_{29}, L_{36}))</td>
</tr>
<tr>
<td>((L_{30}, L_{31}))</td>
<td>((L_{32}, L_{51}))</td>
<td>((L_{34}, L_{48}))</td>
<td>((L_{39}, L_{52}))</td>
<td>((L_{47}, L_{54}))</td>
</tr>
</tbody>
</table>

The normalization process is done by dividing the coefficients by a fixed number, \( N \). This fixed number is analogous to the index number in traditional MDLVQ. In order to reverse the normalization effect, the recovered wavelet coefficients are multiplied by the index \( N \) before applying the inverse wavelet transformation. The normalization process decreases the degree of precision of the wavelet coefficients because some of the details are lost but enables the encoders to cope with the transmission rate restriction of \([0.25 1]\) bpp.

### 6. Experimental results

The performance of the proposed scheme for standard test images of Barbara (512 \( \times \) 512) and Lenna (512 \( \times \) 512) are provided in Table 2. The same experimental setup for the MDLVQ scheme in [3] and [5] is used.

**Table 2 PSNR values for MDLVQ-A4 and previous works in [5] and [7]**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Side1</td>
<td>L1</td>
<td>L121</td>
<td>central</td>
</tr>
<tr>
<td>7</td>
<td>44.88</td>
<td>44.96</td>
<td>44.97</td>
</tr>
<tr>
<td>13</td>
<td>40.43</td>
<td>40.47</td>
<td>40.47</td>
</tr>
<tr>
<td>19</td>
<td>37.96</td>
<td>37.98</td>
<td>37.99</td>
</tr>
<tr>
<td>31</td>
<td>34.97</td>
<td>34.99</td>
<td>34.99</td>
</tr>
<tr>
<td>Lenna</td>
<td>Side2</td>
<td>L1</td>
<td>L121</td>
</tr>
<tr>
<td>7</td>
<td>43.71</td>
<td>43.80</td>
<td>43.81</td>
</tr>
<tr>
<td>13</td>
<td>39.03</td>
<td>39.07</td>
<td>39.08</td>
</tr>
<tr>
<td>31</td>
<td>33.01</td>
<td>33.03</td>
<td>33.03</td>
</tr>
</tbody>
</table>

In Table 2 it is shown that the performance of the side decoders has improved average 6dB and the performance of the central decoder is improved average 1dB.

### 6. Conclusion

In this paper a new **Multiple Description Lattice Vector Quantization** scheme, MDLVQ-A4, is proposed. In MDLVQ-A4 scheme the input stream is quantized by different coinciding lattice quantizers and the labelling function is eliminated.

The advantage of using multiple lattice quantizers in MDLVQ-A4 is that they can represent the input data with different descriptions that can serve as good approximations of the original data. In addition, the central decoder in this scheme finds an approximation for the original data rather that the quantized data. Elimination of the labeling function results in lower computational overhead but larger codebook size. Therefore, the wavelet coefficients are normalized to decrease the size of the codebook. Experimental results for the standard test images of Lena and Barbara show by average 6dB increases for side decoders and 1dB increase for the central decoder compared to [3] and [5].